

**Review of Mathematical Soundness**  
**Background Notes for WA State Curriculum Study**  
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A few standards have been chosen for close scrutiny of their mathematical development. There is no time to do this with whole programs, so a selection of threads has been chosen for this purpose. From a mathematical point of view, the year a standard is learned is not necessarily important as long as the mathematics is developed properly. However, year that standards are covered is noted. A core of the thread for has been carefully selected for review. There are sometimes many other supporting standards that are not incorporated.

**Elementary School (K-5)**

**Whole Number Multiplication**

The standard algorithm for multiplication has particular importance in mathematics for several reasons. First, it is fairly easy to link it to the all-important place value system. Second, it is necessary in order to extend multiplication to decimals; the same algorithm is used, it is just the decimal place that has to be understood. Third, whole number multiplication is used for all four arithmetic operations with fractions:  $\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$ ,  $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ , and  $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$ . Fourth, multiplication is used in the computation of areas for rectangles, parallelograms and triangles, a thread that we will look at next. Fifth, the standard algorithm solves the universal problem of whole number multiplication, taking the *ad hoc* out of multiplication. As such, it is one of the truly beautiful and powerful mathematical theorems that students can learn about in elementary school. Finally, algorithms are important procedures to master for those who might go on into any field that uses mathematics or computers. This is a good place to start the rigorous study of simple algorithms. Learning the algorithm is much more than just learning a way to multiply, it is learning a major mathematical structure.

The thread we consider starts with:

3.2.D *Apply and explain strategies to compute multiplication facts to 10x10 and the related division facts.*

This is an important preliminary, as is indicated in the Washington standards, to:

4.1.A *Quickly recall multiplication facts through 10x10 and the related division facts.*

The standard algorithm is based on distributivity, commutativity, and these single digit multiplications. Knowing all without hesitation is important for fluency with the standard algorithm. Once these are all under control it is possible to connect multiplication with place value, as is expected in:

*4.1.C Represent multiplication of a two-digit number by a two-digit number with place value models.*

The usual representation is to break up numbers using expanded notation and use the area model of multiplication. This gives a nice visual image that connects multiplication to its foundation in the place value system. Other models would also be acceptable, and, in particular, a numerical model is essential to make the step from a representation to numbers. In the numerical model we need:

*4.1.D Multiply by 10, 100, and 1,000.*

Then, multiplication, as it is represented using place value, must be connected to the notation for the standard algorithm so that we can achieve:

*4.1.F Fluently and accurately multiply up to a three-digit number by one- and two-digit numbers using the standard algorithm.*

There is a lot that is important here. The notation for the standard algorithm must be connected to a place value representation so the notation will not seem obscure. The standard is explicit that the algorithm should be learned fluently. This standard will be emphasized as the goal, because without it, even with the foundation of good developmental standards, a student is unprepared to go on in the study of mathematics. Obviously, learning this standard by rote is inadequate, thus the supporting preliminary standards. Once fluent, multiplication should cease to be a problem but should just be a skill that can be used in problem solving. Thus, ultimately, we need:

*4.1.I Solve single- and multi-step word problems involving multi-digit multiplication and verify the solutions.*

Solving problems is the point of mathematics, so this standard is essential. The point of this thread is to develop an understanding of and a facility with the standard algorithm for multiplication so that it can be readily used as a tool for problem solving. This is of fundamental mathematical importance, and this is reflected by its place in the Washington state standards.

### **Whole number multiplication in TERC Investigations**

Investigations Grade 4, Units 1, 3 and 8, all deal with strategies, 3.2.D, leading up to fluency with multiplication facts through  $12 \times 12$ . A great deal of time is spent on this. It is one of the strengths of Investigations: a thorough understanding of small numbers. In addition, in Grade 3, Unit 5 deals with multiplication and develops fluency up to  $5 \times 5$ . 4.1.A is appropriately covered in

Grade 4 as it should be, but 3.2.D is not quite done completely in grade 3, but depends on Grade 4 as well, but is done extremely well there.

Multiplication by 10 is included in the above but 4.1.D is not completed with multiplication by 100 and 1,000.

In Grade 4, Unit 8, page 43, in a teacher led discussion,  $42 \times 38$  is represented using place value and the area model for multiplication. The same is done again on page 57 for  $43 \times 65$ . The numbers are not added up in either case though. Mostly this unit does two-digit multiplication by breaking numbers down, but not in any systematic way. An example of student work that does carry this through to adding up the numbers is shown on page 71, but, again, the solutions expected from the guiding calculations in the student activity book, on page 25 for example, do not lead the student to mimic the place value approach to multiplication.

So, although a good place value representation using the area model of multiplication is included in teacher discussions in Unit 8, there is no priority given to this numerical model. It is offered as just one of many ways to break down numbers in order to find the product. This is not, mathematically speaking, a lead-in to a multiplication algorithm yet, but is still training to find strategies, or *ad hoc*, methods for multiplication.

Standards 4.1.C-D-F-I are not covered in the fourth grade in TERC Investigations. 4.1.C is introduced, but the representation is not solidified numerically. Investigations states (Grade 5, Unit 1, page 71), "It is expected that students enter Grade 5 with at least one efficient strategy for multiplying large numbers." None of the strategies in Grade 4 can be called efficient.

Unit 1 of Grade 5 continues the use of "strategies" for multiplication. On pages 82-83, a nice area model representation of  $35 \times 28$  is given in two ways: where both numbers are broken down according to place value and where just one is. This is associated with the numerical representations as well. This is the first step towards an algorithm. This covers 4.1.C. Soon after, on pages 86-87, multiplying by multiplies of 100 is discussed, nearly covering 4.1.D.

After this brief flirtation with the foundations of algorithms, the rest of the Unit is devoted to practicing more strategies. Even when a place value break down is used for multiplication, such as on page 102, an alternative approach to the problem is given. In this case using  $54 \times 48 = 54 \times (50 - 2)$ . The "strategy" approach to multiplication continues in Unit 7.

A systematic approach to multiplication is addressed in Grade 5, Unit 7, Session 2.3, pages 58-61, where a one-hour class is devoted to the introduction of two algorithms for multiplication. The multiplication  $45 \times 36$  is demonstrated for the standard algorithm (called the U.S. algorithm) and for an algorithm they call the "breaking apart by place," which has been used before as one of several strategies for multiplication (it just gives all four of the partial products and adds

them up). This particular strategy was given an area representation earlier in Unit 1.

Next, students are given pages 21 and 22 of the Student Activity Book. On page 21,  $142 \times 36$  is demonstrated for both algorithms. Students work two problems using both algorithms,  $138 \times 24$  from page 21 and  $184 \times 61$  from page 22. The standard algorithm is not identified as preferred.

On page 36 of the Grade 5 Student Math Handbook, *Investigations* connects the notation for the partial product algorithm to that for the standard algorithm.

*Investigations* states that “Each student may settle on one strategy for each operation that they use most often for routine problems.” Given this philosophy and that this is nearly the final work in the program with multiplication, there is no time to develop fluency with the standard algorithm.

In Session 4 of Unit 7 there is a long sequence of somewhat related problems, some of which are multiplication problems. There are a total of six multiplication exercises in which both factors have more than one digit and are not divisible by 10. In addition, there are about six word problems involving similar multiplications. Again, this is not sufficient practice to develop fluency even if the standard algorithm is always used and it is not clear that the standard algorithm is required. Still the word problems contribute to 4.1.I.

An example word problem is (Student Activity Book page 67):

A store orders 75 cases of juice. Each case holds 24 cans of juice. How many cans of juice will be delivered?

The only example of a multi-step word problems involves division as well (Student Activity Book page 61):

You and 8 friends wash 57 cars. Suppose that you charge \$12 per car. How much money will you earn? If you share what you earn with your 8 friends, how much money will each person get?

This is a nice problem, but badly worded. Sometimes “you” is singular and refers to you, and sometimes “you” is plural and refers to you and your 8 friends.

Before we conclude, there is a teacher-class dialogue on page 146 of Unit 7. In it we have:

This U.S. algorithm was created when all calculation was done by hand. There were no calculators and no computers, so people invented algorithms that had as few steps as possible and that they hoped would be accurate and efficient.

As an historical note, what they call the U.S. algorithm is in the first printed arithmetic book, printed in 1478, somewhat before the discovery of America. This is a math review of the program, so it is important to point out that the use of

“hoped” above is mathematically completely inappropriate. Mathematicians do not “hope” that algorithms work, it is a theorem that they work.

In addition to the standard program there are four sheets of supplements relevant to this thread. In Grade 3, Activity 33 demonstrates single-digit times 2-, 3-, and 4-digit numbers with place value representations. They do not introduce the standard algorithm for this.

In Grade 4, supplementary Activity 60 is on the U.S. algorithm for multiplication. It has a good place value area model for multiplication with the numbers added up. This algorithm is then related, nicely, to the notation for the standard algorithm. Six two-digit by two-digit multiplication exercises are then given.

Mathematically, this does more than move the Grade 5 work with the standard algorithm into Grade 4. It solves some of the connectivity issues by putting the place value area model next to the algorithms and it connects the notations for the two algorithms nicely. There is, however, no lead up to this using single-digit by 2- or 3-digit multiplications with the standard algorithm. There also is no area representation for the standard algorithm.

The main problem is that this activity is a dead end activity; the standard algorithm is not incorporated into the program. It takes more than these nicely done lessons to lead students to fluency.

### Conclusions

The strength of the TERC Investigations in this thread is the development of strategies for single digit multiplication leading to fluency.

The primary problem with whole number multiplication in TERC Investigations is the lack of opportunity to develop fluency with the standard algorithm. The algorithm and a limited number of word problems are included, but there is insufficient time to practice. Significant time is spent developing strategies for multiplication that are strategies germane to numbers in specific problems. This leaves too little time to master the generalized algorithm.

Although a supplementary activity does a solid introduction to the standard algorithm, relating it to the partial products algorithm and the place value area representation for multiplication, it is a standalone activity that is not mathematically incorporated into the program in any way. The main program continues to develop strategies, unaware that the standard algorithm has been developed. The main program does not work to develop fluency with the standard algorithm.

Because they do not develop fluency with the standard algorithm and use this skill to solve multi-step word problems, students are not prepared for the next stage of their mathematical education. Although the explanation of how the standard algorithm works is there in a supplementary activity in Grade 4 and again on one day in Grade 5, the lack of many examples and of time on task

suggests it would be very hard to for students to internalize a real understanding of how the algorithm works.

### **Whole number multiplication in Math Expressions**

Multiplication in *Math Expressions* starts in Grade 2 with array models and basic facts for numbers 2-5. In Grade 3, units 7 and 9 are devoted to multiplication and division. In *Math Expressions* multiplication and division are always developed together through the grades. Mathematically this is both appropriate and attractive.

Unit 7 is concerned with developing strategies and fluency with single digit multiplication where one of the digits is in the range 2-5 and 9. Strategies include repeated group drawings, equal share drawing, arrays, breaking up numbers and using distributivity, and the area model. Practice is included and, from page 197 of the Student Activities Book, it can be seen that students are expected to study multiplications 5 minutes every night. Single-digits are multiplied by 10, 100, and 1,000. Unit 9 extends these strategies and fluency to 6, 7, and 8. The standard 3.2.D is certainly covered.

Underlying Grade 4, as part of the introduction, is a “Basic Facts Fluency Plan.” Serious attention is given to making sure that 4.1.A is met. This includes more work with strategies, review and practice. This continues throughout the year.

The structure of mathematics is part of the content of mathematics, but it is somewhat difficult to capture in standards. The structure of the development of the standard algorithm in *Math Expressions* is gradual and stays on target. Whole number multiplication is fully developed in Unit 5 of Grade 4. There is even a warning that students who have not mastered the single-digit multiplication facts need to continue to work on them.

Once single-digit multiplication has been mastered, the next step is multiplying a two-digit number by a single-digit number. On page 231 of the Student Activities Book (SAB), students are taught the connection between the area representation of multiplication and two different numerical notations where the two-digit number has been broken down by place value. On page 233 a sequence of 5 different numerical notations for this are shown as they evolve from the most basic to the standard algorithm. The standard algorithm (called the “short cut” method by *Expressions*) is then analyzed further.

Next, the place value area model for a single-digit multiplication times a three-digit number is given with three different numerical notations for that explain the mathematics, including the explicit use of distributivity. On page 237, the area model is reproduced with a sequence of five numerical notations for this multiplication as it evolves from the most basic to the standard algorithm. The connections are clear and explicit. On page 238, a detailed analysis of the standard algorithm for this problem is done. The program extends to a four-digit number is used, but this is outside the range of Washington’s standards.

On page 245, there are models for double-digit multiplication with two representations for  $24 \times 37$  given. One is on grid paper (called the rectangle sections method) and one is the area model broken down by place value so you can see the area is the sum of the areas of four smaller rectangles.

On page 247,  $43 \times 67$  is demonstrated using the area model with both numbers broken down by place value and three different numerical notations for what is happening. The standard algorithm is singled out for a detailed step-by-step demonstration and discussion. An area model for it is also given where only one of the numbers is broken down by place value.

Word problems follow, including multi-step problems like (page 251):

Brian's Bike Shop sponsors a cross-country race every summer. Every rider gets an official race T-shirt. The first year of the race, 24 riders competed. Last year, 13 times as many riders competed. If T-shirts come in boxes of 100, how many boxes of T-shirts did Brian need to have for the race last year?

The next few pages, 253-256, involve multiplying by 10, 100, and 1,000, and then numbers like 30, 300, and 3,000.

There are not many problems that use multi-digit multiplication here and although all the standards are otherwise covered, two-digit by three-digit numbers are not included.

Grade 5 *Expressions* begins with students being checked for fluency with single-digit number facts and if they are not ready, they get ready. From page 11 of the Teacher Edition for Grade 5:

“As a Grade 5 teacher, you are aware that some students are not fluent with multiplication and division facts. Unless you bring all students to a competent level, your teaching of multiplication and division algorithms as well as fraction concepts will be hampered by students who struggle to remember facts rather than concentrate on learning new mathematics.”

Next there is a plan to help bring all students to a competent level of fluency.

Whole number multiplication is next addressed in Unit 7 of Grade 5, which starts with a review of multiplication by 10, 100 and 1,000. This is relevant to the standard algorithm, but is also preparation for the decimal work that is coming.

The unit includes a review of what has been covered in previous grades related to multiplication that is enhanced with some new material. *Expressions* gives the area representation, page 266 (SAB), for a two-digit by two-digit example broken down into place value notation, now called the “expanded notation.” Two numeric notations are given beside this area representation. Next, a notation they call “rectangle rows” is given. This is a different precursor notation to the standard algorithm and is, perhaps, a little closer to it. On page 268, another

detailed step-by-step description of the standard algorithm is done. This is all consolidated on page 269 with a two-digit by two-digit example. On this page we have the “rectangular sections” (just the area representation with both numbers in expanded notation) and the numerical version of expanded notation for this multiplication. Also, we have the area representation for the rectangle rows and the standard algorithm (the representation is the same) along with their respective notations. The connections among all of the representation is on the same page and exceedingly clear. On page 271, the same thing is done with a three-digit by three-digit multiplication. At this stage the efficiency of the standard algorithm is glaringly obvious.

Word problems using multiplication are interspersed. There are about a dozen of them. Most problems are single-step problems but combined they can become multi-step. An example:

Farmer Ruben’s rectangular wheat field is 789 meters by 854 meters.  
What is the area of this wheat field?

There are also lists of exercises.

On page 659 of the Teacher Edition there is something that gives a brief moment of alarm where the Washington state standards are concerned because they let students chose what method to use for some problems. In fact, all of their mathematics leads directly to the standard algorithm, and, the next thing they have are suggestions for how to help struggling students in states that require the standard algorithm.

### Conclusions

With great clarity and simplicity the program goes from a geometric representation of multiplication using place value through a sequence of notations to end up with the standard algorithm for whole number multiplication. This is elegantly done. Although the content is spread out over three years, each time it comes up the previous material is reviewed, or, really, redone, so that the thread is not lost.

All of the standards in the thread are covered and connected to each other. However, despite the dozen or so word problems in Grade 5 and others in Grade 4, there can never be too many word problems. There are certainly not enough multi-step problems.

Students will be well prepared to go on to the next level.

### **Whole number multiplication in Bridges in Mathematics**

Grade 3, Volume Two, Unit 4, covers 3.2.D very well.



In Grade 4, Volume One, the goal is fluency and there are practice units throughout, mostly in the Number Corner, for 4.1.A.

Unit 2 of Volume One of Grade 4 focuses on multiplication, but student proficiency is limited to one-digit by two- and three-digit multiplication (page 145) in the main program. In Activity 12 of main Bridges, area models with place value are used to represent both the partial product algorithm and the standard algorithm, side by side, and then there are more examples. It is important to note that after each incremental instructional step there are good word problems. There is, however, a shortage of instruction and examples of two-digit by three-digit multiplications. We count only three such numerical problems and one word problem.

At the very beginning of Unit 1, pages 5-6, there are some very nice area models for commutativity and distributivity. It is not clear whether these make it into student materials or not.

In Unit 2, for example on pages 143 and 146, there are good area place value models for two-digit multiplications. In the second case the computation is also illustrated with numbers. In fact, the standard algorithm notation is there with no explanation except to say that “it’s possible for students to see and understand each step of the traditional multiplication algorithm.” No help is given.

Multiplication by 10 is done in Session 6, Unit 2. In Session 7, place value arrays are given for multiplication of two-digit numbers, including practice problems. Place value array models continue to be developed in Sessions 8-11.

The standard algorithm is brought up on page 220 for a one-digit by two-digit example. It is not demonstrated though. Instead, the partial products algorithm is demonstrated as “easier to follow.” Teachers are instructed to connect the standard algorithm in this case to array models.

At the end of Grade 4, using their basic materials, students should be fluent with the single digit multiplication facts, know how to multiply by 10, and have worked with array models of multiplication of two-digit numbers that use place value. They have worked with single-digit by two-digit multiplications but algorithms have not been developed although they have been introduced.

However, in the *Bridges* supplement for Washington, the rest of whole number multiplication is addressed. Standard 4.1.D is neatly taken care of first. Next, single-digit numbers are multiplied by two-digit numbers. Area models using place value are used. Partial products are written beside the standard algorithm. The area model in this case is the same. There are helpful hints as to how to make the transition from the partial product algorithm to the standard algorithm. In Activity 9, the usual place value area representation of multiplication for two-digit numbers is given along with the partial product algorithm. On page A5.92 of Activity 11, there is a transparency with a nice sequence for a two-digit multiplication. It starts with the area model using place value and adding up the

four numbers, moves to the partial product algorithm, and then to an expanded version of the standard algorithm. Finally, the standard algorithm finishes it up. This shows well the evolution of the various computation notations. More examples are worked.

Moving on to Grade 5, in Unit 2, the area model for multiplication is used again. However, it is not always done by place value. For example, on page 158,  $23 \times 12$  is modeled but 23 is not broken up by place value, i.e.  $23=20+3$ , but as  $23=10+10+1+1+1$ , as were many of the Grade 4 models. This issue is carried along into Sessions 3 through 7, but in Sessions 9-10 the area model is done nicely for two-digit multiplication using place value, consolidating the previous Sessions. In addition, these models are connected up nicely enough with the partial products algorithm although the numerical version that demonstrates the use of commutativity and distributivity is not done.

In Session 10 the standard algorithm using two-digit numbers is demonstrated. In discussion this is connected to an area model, but the extrapolation from the notation of the partial products algorithm to this is not made clear. There are practice problems with the standard algorithm.

Session 11 reviews multiplication and does a nice consolidation of approaches, all on one transparency. In particular, without identifying it by name, there is an example of the standard algorithm beside an area model for it that uses place value.

The standard algorithm is never singled out as anything special or more important than other “strategies.” Session 12 presents an extended word problem that includes numerous multiplications. This is a very nice multi-step word problem, but in the example solutions, the standard algorithm is only used once, and another, non-algorithmic option is shown next to it. Instructing students to use the standard algorithm could easily change this. There is homework with larger numbers, but no examples are worked.

In Session 10 there is a problem with the use of mathematical language that is not restricted to this session but is illustrated well here. On page 234 we have “All strategies, including the standard algorithm, have their limitations.” We were unable to find a definition of “strategies” in the materials. Mathematically speaking, the standard algorithm would not be considered a strategy for multiplication, on the contrary, it is the solution to all possible multiplication problems. So, it is unclear what is meant by “limitations.” Most of the other approaches in *Bridges* are indeed strategies. Students learn to take numbers apart and figure out how to multiply them, but because the way this is done depends on the numbers they aren’t algorithms, but strategies. It is mathematically incorrect to equate analyzing numbers in this way to the standard algorithm by calling both approaches strategies. The difference is significant and the terminology used here muddles the mathematics. Learning about the standard algorithm is to learn about a major mathematical structure, and such

structure is a major part of mathematics. This is several levels above learning “strategies.” This is not just about learning how to multiply.

This de-valuation of the standard algorithm continues in Session 4, page 201, where three simple approaches to multiplying 14 by 12 are given and then four versions of “mis-memorized or mis-applied algorithms” are also demonstrated. Surely there is pedagogical value to this, but by the same token, properly applied and understood algorithms are more important than the approaches that are done correctly.

### Conclusions

The original program finishes most of the thread in Grade 5. A numerical model that demonstrates the use of commutativity and distributivity is not done and three-digit numbers are not used. The connections are done nicely enough but not always explicitly.

In the Washington state supplement for Grade 4 the thread is mostly covered. Again, the numerical model using commutativity and distributivity is not done and there is minimal work with three-digit numbers. The transitions from representations to partial products to the standard algorithm are handled nicely.

Mathematically, the program does not celebrate the standard algorithm, but seems to always be looking for alternative ways to do things rather than use the standard algorithm. This is not an attitude that is conducive to learning this important mathematical structure.

### **Whole number multiplication in Math Connects**

Elementary multiplication strategies are begun in Grade 2, Chapter 15. These concepts are carried through in Chapters 4 and 5 of Grade 3. In Chapter 15 of Grade 3 two- and three-digit numbers are multiplied by one-digit numbers. There are place value area models for the two-digit by one-digit multiplications which are presented side-by-side with the partial products algorithm and the standard algorithm, giving a clear connection between them. There are also many detailed examples of two- and three-digit numbers multiplied by one-digit numbers using the standard algorithm. These examples are spelled out in great detail. In some cases very geometric representations of the multiplication are included so that the regrouping done in the standard algorithm is explained and demonstrated carefully.

From Grade 4, Chapter 6, it is clear that 4.1.A is established. Multiplication is done by single-digit numbers times single-digit multiples of 10, 100, and 1,000. This is not quite 4.1.D, but close. Examples are done of two-digit times one-digit numbers using place value, distributivity, partial products, area with place value, and details of the standard algorithm. Going further, four-digit numbers are multiplied by one-digit numbers using the standard algorithm explained in great detail along with a place value area representation and the partial product version. An unusual, but very nice, side trip is taken to explain what happens if

there is a zero in a number, such as 108. This is done with place value, partial products and the standard algorithm.

In Chapter 7 there are several examples of two-digit by two-digit multiplications with area representations that use place value, all on the same page with the partial product approach and a very detailed version of the standard algorithm. More examples are given and eventually the partial product version is dropped. Detailed three-digit by two-digit multiplications are carried out for the standard algorithm with the area place value representation right beside it.

There are numerous word problems in Chapter 7, for example (page 287):

A person breathes 95 gallons of air every hour. How many gallons of air does a person breathe in one day?

There are even multi-step problems, such as (on page 286):

The average number of hot dogs eaten in one year is 60 and the average number of slices of pizza is 46. How many more hot dogs than pizza slices will a person eat in 15 years?

This is a nice multi-step problem but not very well posed since you cannot compute “exactly” how many from the “average.”

In Grade 5, Chapter 3, there is an analysis of what happens with zeros at the end of a number under the operation of multiplication. This is not quite 4.1.D, but it does seem to include it. There is also more about multiplication in this Chapter.

### Conclusion

This thread is nicely covered except for the lack of a numerical model for multiplication that contains place value and demonstrates the necessary commutativity and distributivity all in one place. On the whole students will be well prepared for the next level of mathematics.

### **Area of a triangle**

Area is a fundamental concept in mathematics. The area of a triangle is derived from that for a parallelogram (by taking two copies of the triangle and making a parallelogram), and the area for that is, in turn, derived from the area of a rectangle (by cutting up the parallelogram and rearranging the pieces to make a rectangle). The area of a rectangle follows from multiplication (also important) and the area of a square, which essentially gives the definition of area.

With the area of rectangles, parallelograms, and triangles under control, the area of polygons and planer surfaces can be calculated. In practice, measurement also comes into play. The computations of volumes frequently depends on knowing the areas connected to the solid. As the student progresses, the techniques for computing area also progress. The Integral Calculus allows

students to extend their knowledge of area to much more complex figures, but depends heavily on the understanding of area from elementary school.

This thread begins with the area part of:

4.3.C *Determine the perimeter and area of a rectangle using formulas, and explain why the formulas work.*

The “explain why the formulas work” part of this is essential. Next, we go to:

5.3.D *Determine the formula for the area of a parallelogram by relating it to the area of a rectangle.*

The goal of this thread is given by the next two standards:

5.3.E *Determine the formula for the area of a triangle by relating it to the area of a parallelogram.*

and

5.3.I *Solve single- and multi-step word problems about the perimeters and areas of quadrilaterals and triangles and verify the solutions.*

### **Area of a triangle in TERC Investigations**

In Grade 3, Unit 4, Investigations works with the meaning of area in terms of unit squares. Pages 29 and 30 of the Student Activities Book give some very nice, not completely straightforward, problems for the computation of areas of rectangles (part of the rectangles are covered). The areas of some very simple triangles constructed on small grids are also computed. It is always easy to see that the area is one-half the area of a rectangle whose area is also immediately determined by the grid. Even though this is Grade 3, a little early for Washington standards, students should be able to compute the areas of rectangles. Formulas, however, are not developed.

Unit 4 of Grade 4 spends some time finding areas in terms of triangles whose areas are not known. This is off the thread we are considering though. The area of arbitrary shapes on a 5-peg-by-5-peg Geoboard are studied. There are severe restrictions on the polygons that can be constructed and their areas are quite easy to compute. More elaborate polygons are then considered. On page 138, area for rectangles is connected to the computation of arrays, i.e. multiplication, but, again, there is no formula, not even an explicit statement of how to compute the general area of a rectangle.

Areas of rectangles are then considered and areas are computed using grid paper, multiplication or skip counting, but formulas are not considered. Some areas of triangles are considered, but only right triangles where the area is obviously half of that of a rectangle that is easy to compute (on a Geoboard).

Most of 4.3.C is done, almost including the explanation of why the formula work, except that there is no formula.

In Grade 4 there are two supplementary lessons—Activity 24 and 25—on the area of a rectangle. The formula for the area is given and a worksheet with problems is there. Since all that was missing from the regular program was the formula, this integrates nicely.

The final work on areas in Investigations is carried out in Grade 5, Unit 5, Investigation 2. There is more work with areas of rectangles and then students learn to draw parallelograms, but the area of a parallelogram is never considered, nor is the area of a triangle considered.

In Grade 5 there is a supplementary Activity 20 about areas of parallelograms and triangles and their formulas. They begin by putting two identical triangles together to make a parallelogram. This shows that the area of a triangle is half that of the parallelogram. This is a truly minimal demonstration with no elaboration. Next they show how to turn a parallelogram into a rectangle and derive the formula for the area of a parallelogram.

This, in turn, gives the formula for the area of a triangle as one-half of that for the parallelogram constructed by two of the same triangle. At this stage there is a big mathematical gap. For the formula for the area of a parallelogram, the height is described by way of a picture of a parallelogram with the height labeled. Nowhere is the height of a triangle defined in any way. This is problematic when it comes to applying the formula.

There is a problem sheet, but with few word problems, and none of them is multi-step. There are also problems of finding the area of figures, but as they are on grid paper, the use of formulas is unnecessary.

### Conclusions

With the help of a one-page supplement, much of this thread is covered. Formulas are derived, but the work for the area of a triangle is both very weak and has a significant gap. There are an inadequate number of challenging problems.

### **Area of a triangle in Math Expressions**

In Grade 2 the concept of the square of unit area is developed. In Grade 3, Unit 8, using square units, the areas of rectangles are computed using multiplication. This leads to the formula for the area of a rectangle.

In Grade 4, Unit 2, starting on page 102 of the Student Activity Book, the area of a rectangle is fully developed (4.3.C). This starts with the unit area and uses grids to make the area of a rectangle clear, including the formula for the area again. This is followed by a collection of word problems.

On page 110 the height of a parallelogram is defined and a parallelogram is cut up and rearranged to give a rectangle with the same base and height, thus giving the connection between the two and their areas. The formula for the area of a parallelogram is then derived on page 111 (5.3.D).

In Grade 4, Unit 4, we see how to build rectangles and parallelograms from two copies of a triangle and how to split rectangles and parallelograms into two congruent triangles along a diagonal. The areas of right triangles are then done on grid paper. Next, triangles are drawn on grid paper in such a way that the altitude divides the triangle into two right triangles so the area can be computed and the formula for the area of a triangle is derived in this way (on page 203 of the SAB).

Near the beginning of Unit 2 of Grade 5, on page 57, we have the formulas for the area of rectangles, parallelograms and triangles, complete with pictures and illustrative examples. This is a summary of what is to be done in this Unit. The area of a square is done, to establish the unit area, and then the area of rectangles is done (yet again), including formulas and word problems. The area is developed with grid paper.

Next, the area of right triangles is developed where it is easy to see that the area is half the area of a rectangle (page 66, SAB). Moving on (page 66A), they “experiment with parallelograms,” finding the way to compute the area by comparing them to rectangles (5.3.D). Next, they find the formula for the area of a parallelogram (page 67). After that, they “experiment with triangles” (page 68A), showing how if you take two the same you can piece them together to get a parallelogram and figure out the area of the triangle from that (page 69) (5.3.E). There are numerous problems, mostly centered on geometric objects (5.3.I).

## Conclusions

The formulas for the areas of a triangles, parallelograms, and rectangles are all developed nicely, and the area of the triangle is related to that of the parallelogram, and this to the area of a rectangle. All of this is done with clarity. The thread is completely covered. It is worth noting that all of the material is developed in the Student Activity Book without the need to reference the Teacher Edition.

## Area of a triangle in Bridges in Mathematics

The serious investigation of area begins in Grade 2, Unit 4. Rather than begin with a definition of area, i.e. with a unit square, unit areas are chosen somewhat arbitrarily. Area is just covering a surface with a unit shape and

“Theoretically, this unit can have any shape: rectangular, triangular, etc.  
The number of units needed to cover a surface is called the area.”

Initially, triangles are used for area. In Session 9, squares are used on Geoboards. By Session 14, what would have worked well as an introduction to

area is given under the heading “If a square is worth 1”. This is immediately followed up in Session 15 by using two different unit areas, a triangle and a rhombus, obscuring the ultimate goal of defining what we mean by area by leaving second graders with multiple ways to compute areas, depending on what shape they chose to use as their unit.

Unit 4, Grade 3, is about multiplication and division and they use the array model frequently. Area seems to come up, but rarely. In the introductory material for the Unit we have a couple of statements related to area, for example, on page 427, “The area of a 4-by-3 array is 12.” There is no discussion of area and what it means and why this area would be 12. The fundamental definition of area, starting with a unit square, seems to be missing. There is no transition from the arbitrary unit shapes used in Grade 2.

The first reference to “area” is found is on page 475. This is still about the array model of multiplication. “Array” is mentioned seven times on this page and then students “need to label its linear dimensions and find the total area. Challenge them to find the area in as many ways as they can ...” It seems that children are to intuit that real area is computed using unit squares, but with no explanation. This is a bad mathematical start to the study of area. Later, on page 495, homework has a game “based on the area model for multiplication.” The “area model” of multiplication never seems to be developed, especially since area has been computed with arbitrary shapes up to this point. (This is constantly qualified because there is no index and without a careful reading of every page, and with thousands of pages in *Bridges*, it is impossible to say for sure what has and what has not been done.

Moving on to Grade 4, Unit 1, Session 6, there is a good definition of area starting with a unit square. This is immediately undermined mathematically by using 2-by-2 squares and 3-by-3 squares to compute the areas of the same regions in terms of the number of different-sized squares. (When using 2-by-2 squares the area is computed in terms of the number of 2-by-2 squares used.) Students get three different answers for the area of the same region. Perhaps if they were using different unit squares this might be good pedagogy (i.e., inches and centimeters) but because the different “unit” squares are clear multiples of the first unit square, this is lost and what area really is becomes unclear.

Area is taken up again in Session 19 under a promising heading “Defining Area and Perimeter.” However, the definition is left up to the student with “students will likely be able to generate working definitions for both words.” In mathematics, definitions are not “discoverable.” If “students have trouble defining the words”, then “Explain that the tile[s] show the area.” Combined with the work in second and third grades, this is an unsatisfactory introduction to area. In Session 20 it is just assumed that the product gives the area. This foundation of area is mathematically unsound. *Bridges in Mathematics* sums up the apparently intentional confusion about area in Unit 4, Session 16, page 493, with



... ask students to share what they think area means. By fourth grade, students will have had some experiences calculating area and are likely to understand area as the covering of space. They may even mention square units, such as square centimeters or square inches.

In *Bridges in Mathematics*, the definition of area is “discovered” by computing areas, but it is not possible to compute areas without a definition of area.

Grade 5 is where the thread should be consolidated. In Unit 2 the area model for multiplication is assumed and used, but this seems to rest on the shaky foundation established in Grades 2, 3 and 4. In Unit 3 things seem to regress significantly. On page 348 under the heading “Discussing how to find the area of any rectangle” we have

Give students some time to read and think about the question and then solicit responses. It is likely that someone will suggest multiplying the length by the width, and you may find that this is a rather brief discussion, given students’ prior experiences using the array model for multiplication. Once students mention something resembling the standard formula, ask how they could verify it.

Given that students know the area model for arrays, the class discuss should lead to the formula, rather than

Some students will immediately adopt the formula (length  $\times$  width = area), but others will prefer counting strategies. This is fine, and the formula will become part of students’ repertoire as they are ready.

Multiplication is mastered in the fourth grade in Washington, so “counting strategies” are not acceptable. The formula is also a fourth grade standard, so it cannot be put off for later when in the fifth grade.

While students, in Unit 3, compute areas of the very limited triangles and parallelograms that can be made on a small Geoboard, the teacher is given a full explanation under the heading “Background for teachers: area formulas for rectangles, triangles, and parallelograms.” This begins with “Although your students will develop their own methods for finding the area of each figure, the information below is provided for your benefit.” Then, in less than a page, the teachers are taught how to find the formula for the area of a parallelogram from that of a rectangle and the formula for the area of a triangle from that of a parallelogram. This is presented beautifully and efficiently. Unfortunately, students are never given this information.

*Bridges in Mathematics* has supplementary materials for Washington State related to this topic.

In Grade 4 and its supplement for 4.3.C there is, in D6, a reasonable definition of area and, a couple of pages later, the formula for the area of a rectangle.

In the Grade 5 supplement C1, *Bridges* does a very odd thing. The formula for the area of a parallelogram is developed by relating it to the area of a rectangle. *Bridges in Mathematics* does exactly the opposite. It takes rectangles and cuts them up to produce parallelograms with the same area. This is done multiple times and then, on page C1.20, the students are given the formula for the area of a parallelogram. This is mathematically backwards. To do this right, we need to cut up a parallelogram and rearrange the pieces to get a rectangle so we can calculate the area. That is not what is done. The formula is used to compute the area of parallelograms.

In 5.3.E, the formula for the area of a triangle is derived from that of a parallelogram using the fact that any two copies of the triangle can be put together to form a parallelogram. This is done using the Geoboard. The problem with this is that there are very few kinds of triangles one can make on a Geoboard. *Bridges* includes only one non-right triangle example and has no applications for area of triangles. (5.3.I).

### Conclusions

The definition of area is never clearly given in the regular program, but it is included in the supplemental materials. In the regular program areas are computed using the area model for multiplication, but without the benefit of such a definition. The special Washington state supplements properly define area and to get at the formula (4.3.C). The derivation of the formula for parallelograms is mathematically incorrect (5.3.D) and the derivation of the formula for the area of a triangle is inadequate because it is done using only the very limited examples from Geoboard triangles (5.3.E). There are no word problems (5.3.I), only computational examples.

### **Area of a triangle in Math Connects**

In Chapter 2 of Grade 2, the definition of area using a unit square is clear: “The number of square units it takes to cover a space.” This is used to compute areas. In Grade 3, Chapter 9, areas are computed on Geoboards using slightly more complex areas than in Grade 2, but still not very complex.

In Grade 4, Chapter 11, the formula for the area of a rectangle is done. In Chapter 14 of Grade 5 this is done again in review. Parallelograms are cut up and rearranged to give rectangles, allowing for the computation of the area of a parallelogram. However, the formula for the area of a parallelogram is not given and, worse, triangles are not even dealt with.

### Conclusion

The definition and the formula for the area of a rectangle are done nicely, but the formula for the area of a parallelogram is not given even though it is shown how to compute the area. The real failure is the lack of anything about the area of a triangle. This thread is incomplete, but what there is, is mathematically sound.

## Fractions and the arithmetic of fractions

The importance of fractions to mathematics cannot be overstated. Numbers in mathematics start with whole numbers and then increases to integers. Next fractions are developed to get the rational numbers, working first with positive numbers before adding negative numbers. From the rational numbers, the real numbers can be constructed, although the actual construction doesn't happen in K-12. For middle and high school students, real numbers are mostly taken for granted, but they still get to see complex numbers. The point is that fractions are an essential intermediary step between whole number and real numbers.

With the real numbers one can put coordinates on the line or the plane, and from this can turn geometry into algebra and algebra into geometry.

The step of going from the whole numbers or integers to fractions is the same step taken to go from polynomials to rational expressions. This is a common transition in mathematics, and these elementary school fractions are the easiest place to start.

It is impossible to overstate the importance of fractions. Numbers and geometry are at the heart of mathematics, and fractions are required for both. You can't do mathematics without an understanding of fractions and their operations.

The four arithmetic operations with fractions are divided between grades 5 and 6 in the Washington state standards. This crosses the boundary between elementary school and middle school and so it is possible that different programs will be used for these two grades. This makes it all the more important to accomplish the grade 5 standards in grade 5. This section focuses on the elementary school topics.

This thread starts with seeing a fraction as a number, and putting it on the number line will do, as in:

*3.3.A Represent fractions that have denominators of 2, 3, 4, 5, 6, 8, 9, 10, and 12 as parts of a whole, parts of a set, and points on the number line.*

Next, there needs to be a definition or a representation of addition and subtraction, as numbers:

*5.2.A Represent addition and subtraction of fractions and mixed numbers using visual and numerical models, and connect the representation to the related equation.*

Finally we need a purely numerical way to add and subtract fractions:

*5.2.E Fluently and accurately add and subtract fractions, including mixed numbers.*

and the fraction part of

5.2.H *Solve single- and multi-step word problems involving addition and subtraction of whole numbers, fractions (including mixed numbers), and decimals, and verify the solutions.*

### **Adding and subtracting fractions in TERC Investigations**

Grade 2, Unit 7, is about fractions and modeling fractions, but does not make the transition to numbers.

Grade 3, Unit 7, is about fractions. The fractions are represented as parts of cookies and brownies and sets in general, and then, the cookies are left off the notation and what remains is a fraction that appears to be a number. Mathematically, this is not sound. The transition from parts of cookies to numbers has not been made. So, on page 75, the issue is whether the equation,  $\frac{1}{6} + \frac{1}{4} = \frac{1}{2}$ , is true or not. Since fractions have not been turned into numbers, this makes no sense. As they discuss it, students are told to think cookies or brownies.

The Student Activity Book, page 23, “Many ways to make a share,” tells students to “Think of sharing brownies or hexagon cookies” and asks students to “write all the fractions you know that work.” Question 1 is “Ways to make 1 whole.” This is okay, because we are still talking about, presumably, 1 whole brownie, but one whole anything would do. But the other questions go down the slippery slope to fractions as numbers, without ever having defined fractions as numbers, for example, problem 3 is “Ways to make  $\frac{1}{4}$ .” The transition from “fractions of things” to “stand alone fractions” must be done more rigorously than this, just dropping the “things” is not mathematics, but this seems to be how the transition to numbers is done in Investigations. After this point, seeing fractional numerical equations is common. This transition is murky and seems to start on about page 72 and be complete by about page 77. This is not an acceptable way to define fractions as numbers. What is needed are definitions that are so clear that students grasp and hold on to them.

After all of this, fractions are turned into numbers, as points on the number line, in a supplementary Activity 71, to be inserted after page 86. Mathematically, the development would have been much better if the option of thinking of fractions as numbers had been introduced in Unit 7, Session 2.1 instead of after Session 2.4. This would increase the likelihood that students understand fractions as numbers.

Having defined fractions as numbers in the Grade 3 supplement, many of the problems of ambiguity in the Grade 4 materials disappear. But there is still some continued lack of clarity. In a note on page 42, there is a mention that  $\frac{1}{2}$  can be represented on the number line: “In one sense,  $\frac{1}{2}$  is always  $\frac{1}{2}$ . When represented on a number line, for example, the number  $\frac{1}{2}$  always has the same relationship to 1 and to other numbers.” They complicate the discussion by then saying “However, the quantity represented by  $\frac{1}{2}$  depends on the size of the

whole quantity.” This discussion causes more confusion than is warranted by the mathematics.

In Session 2.5 of Unit 6 of Grade 4, the number line is introduced again. Students put the fractions they know about on the number line. It is not clear how this is done as there is no explanation, and to put them on the number line requires some input. The Grade 3 supplement offers guidance that is missing in the regular program.

The professional development section clarifies what information is given to teachers. Page 140 states, “one interpretation of  $\frac{1}{2}$  is that it represents one out of two equal parts of a whole, but it also means the quantity that results from dividing 1 by 2.” This shows the mathematical misconceptions in the program by suggesting that  $\frac{1}{2}$  can mean two different things. Better would be to take the number line and define the number  $\frac{1}{2}$  as the point  $\frac{1}{2}$  of the way between 0 and 1, i.e. taking 1 to be the “whole” and taking  $\frac{1}{2}$  of it. With this, the two “meanings” of  $\frac{1}{2}$  can be reconciled, since  $\frac{1}{2}$  should not mean two different things.

Overall the foundational work with fractions in grades 3 and 4 is both weak and confusing. The transition to fractions as numbers is accomplished only with the help of a one-page supplement in Grade 3. The program’s ambiguous sense of what fractions mean is confusing.

In Grade 5, Unit 4, Session 3.1 models the addition of fractions using an analogue clock face. Using a clock to model fractions and the addition of fractions works nicely for many common fractions because it can handle fractions whose denominators divide 60. Numerical models are needed to, among other things, define addition of fractions as a number. The clock is a nice visual representation, but it does not qualify as such a definition. In addition to the clock representation, *Investigations* uses rectangles with grids to give another visual representation of fractions and their addition and subtraction. This is another good representation, but again, is not a representation of addition and subtraction as numbers.

Starting in Session 3.4 of Unit 4 of Grade 5, *Investigations* introduces the number line to define fractions as numbers and does it well enough. It introduces the “Fraction Track Game” which introduces adding fractions as numbers on the number line. Using a number line is an excellent approach. *Investigations* method is slightly problematic mathematically because it uses different lines for numbers with different denominators, when, from a mathematical point of view, all of these numbers exist on the same number line. Additionally, the initial representation is all about fractions between 0 and 1 although the program moves on to fractions between 0 and 2 later. This reinforces the sense that fractions are all “little numbers”.

Fractions are added and subtracted using very concrete representations of the fractions: rectangles with grids, clocks, and fraction tracks (the number line). If a student were to be given a simple problem, simple, that is, if you know about

common denominators, such as  $\frac{1}{7} - \frac{1}{9} = \frac{9}{7 \times 9} - \frac{7}{7 \times 9} = \frac{2}{63}$ , they could not use the clock approach since the denominators do not divide 60. They could not use the number line because dividing the unit interval up into 63 parts is a bit unwieldy. Students could use the rectangles with grids if given the right size rectangle, but they are not taught how to find the right size rectangle in order to solve a problem like this. Even if they knew how to find the right size rectangle, this approach would get unwieldy very fast with larger denominators.

Adding and subtracting arbitrary fractions requires the use of common denominators, but this is not developed well in *Investigations*. The only mention found in grade 5 is on page 100 of Unit 4 (Session 3.1) where there is a math note in the upper left hand corner that states: “One useful strategy students encounter in later grades for adding and subtracting fractions is finding equivalent fractions with a common denominator.”

In two supplemental activities to be added to the program after Session 3.10 of Unit 5, the least common multiple and adding and subtracting fractions are discussed. There is, in the second supplement (Activity 7), the statement “One way to find a common denominator is to multiply the denominators.” No examples are worked and no problems needing this approach are given. This approach is never mentioned again. The very next statement is “However, sometimes there is a smaller common denominator.” This is the emphasis and it is the subject of the first activity on least common multiples. Least common multiples are computed by writing down all the multiples of the two numbers until the least common multiple is found. There are better ways to do this mathematically. The *Investigations* approach runs into serious problems with numbers like 17 and 19. However, in the problem section, the hardest problem is to find the least common multiple of 5 and 8, and students are given a number chart to 100 to help them out. Furthermore, when students do fraction addition and subtraction problems, the hardest example is  $\frac{1}{3} + \frac{2}{5}$ .

There is no systematic development of common denominators that can be applied to arbitrary fractions, and no applications that would require such a development.

5.2.E is a prerequisite for the fraction part of 5.2.H. Although there are word problems involving the addition and subtraction of fractions, they require the thinking about the adding and subtracting of fractions as well as the word problem. 5.2.H is there to use the skill of adding and subtracting fractions to solve word problems, not to teach “strategies” for adding and subtracting fractions, as *Investigations* likes to say, and as they say on page 10, Unit 4 of Grade 5: This Unit is, among other things, for “the development of strategies for adding and subtracting fractions.” “Strategies” for adding and subtracting fractions do not fulfill 5.2.E. The use of common denominators is not a strategy: it is the solution to the addition and subtraction of arbitrary fractions.

## Conclusions

The representation of fractions and their addition and subtraction, through representations, is nicely done in *Investigations* with their clock model and their rectangular grid model. This only happens after the weak and confusing foundational work in Grades 3 and 4. *Investigations* fails to prepare students to work with common fractions with denominators not related to 60 and with arbitrary fractions using common denominators. Its work with mixed numbers is extremely limited.

Students using this program will not be well prepared to go on in mathematics. A great deal of additional supplementation is needed before a typical student would meet the standards associated with this thread.

### **Adding and subtracting fractions in Math Expressions**

Grade 3, Unit 11, is about fractions. This examination looks carefully at fractions as numbers. On page 419 of the Student Activity Book (SAB), fraction strips are introduced to discuss equivalent fractions and simplifying fractions. Fraction strips are then used to define the addition of fractions (page 431). This is the same as using the number line except that *Expressions* keeps their fractions and sums below 1. It also does subtraction (page 437) in the same way. The number line is introduced on page 439, and to make it clear that it is not just between 0 and 1, the first version goes out to 500. The line is shrunk down to compare with fraction strips and move fractions to the number line. By page 448 *Expressions* is asking students to put mixed numbers on the number line, again making it clear that fractions are more than just numbers between 0 and 1. In the Unit Test there are some simple addition and subtraction problems that involve fractions that do not have the same denominators, for example  $\frac{2}{3} - \frac{4}{9}$ . This more than accomplishes 3.3.A but also 5.2.A.

Grade 4, Unit 9, continues the study of fractions. Most fractions are depicted on fraction strips. This is not as nice as the number line representation because fractions with different denominators are represented on different strips. They can be placed over each other for comparison, but it is not as elegant as using the number line. The definition of addition as concatenation is used and computations are also done numerically.

On page 378 (SAB) common denominators are introduced to “add unlike fractions.” Least common denominator is also introduced. Many exercises are given with addition of unlike fractions but a systematic way to find a common denominator is not apparent in the SAB. Students, at this stage, have had lots of practice with equivalent fractions and the exercises start out easy and gradually get harder, to, for example,  $\frac{5}{8} - \frac{4}{7}$ .

The method given by the formula,  $\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$ , is not in the SAB. Certainly the denominator here is sometimes bigger than necessary, but it always works. On page 925 of the Teacher Edition it explains

To find a common denominator, multiply the numerator and denominator of each fraction by the denominator of the other fraction.

Mixed numbers are also dealt with in Grade 4.

Unit 5 of Grade 5 is titled “Addition and subtraction with fractions.” It starts with a family letter explaining the goals and gives the formula for adding and subtracting fractions with like denominators along with numerical examples. It then explains that they will extend these skills to fractions with unlike denominators by finding common denominators. Finally, the goal is to deal with mixed numbers.

Fraction bars and the number line are used to define fractions as numbers. Fractions with like denominators are dealt with easily enough and there are numerous word problems in addition to some very nice equation problems such as  $\frac{2}{5} + \frac{n}{d} = 1$ , on page 187 (SAB). A typical word problem, on page 188, is

We ate  $\frac{5}{12}$  of a watermelon at breakfast. At lunch we ate  $\frac{7}{12}$  of the same melon. How much of the watermelon is left for dinner?

There are also some much longer, much more sophisticated problems, such as

The 4 runners on a relay team all want to run the same distance in a race. The coach says that the first runner will go  $\frac{1}{4}$  of the distance. Then the second runner will go  $\frac{1}{3}$  of the remaining distance. The third runner will go  $\frac{1}{2}$  of the distance that is left at that point. The fourth runner will finish the race. Will each runner run the same distance? Why or why not?

For fractions with the same denominator, fraction bars work nicely to represent addition and subtraction of fractions. Unlike the number line, fractions with different denominators are not represented on the same bar, but bars can be stacked on top of each other for comparison. All fractions live on the number line though.

The development moves on to a long section on equivalent fractions from several points of view. The representations are very nice.

Next, students are told to find common denominators to add and subtract fractions. They are not, unfortunately, told how to do this in the SAB, but they learned how in Grade 4. This is illustrated with fraction bars. It is expected that something more than fraction bars are understood because they ask students to work problems like  $\frac{3}{11} + \frac{1}{2}$  and  $2\frac{1}{6} + 1\frac{1}{9}$ . Although the student text lacks explicit instruction as to how to find common denominators, the Teacher Edition, on and



around page 512, goes into some detail. In particular, it explains that the easiest, most straightforward way, is to just multiply the two denominators. They appear to be hesitant to emphasize this because of the many examples where this number is much larger than it needs to be. The advantage, of course, is that it always works. There is a brief discussion of the least common denominator as well.

### Conclusions

Math Expressions does an excellent job of developing the addition and subtraction of fractions. Fractions are defined as numbers using fraction strips and the number line and addition and subtraction are represented on the number line as well. The student texts are a little weak on finding common denominators but the Teacher Edition discusses it thoroughly. There are good problems. This thread is covered and students should move on prepared.

### **Adding and subtracting fractions in Bridges in Mathematics**

Significant work is done with fractions in Grade 3, Unit 6, but it is always with fractions “of” something like pizzas not fractions as numbers. Washington standards require that students move to getting fractions as numbers. In Session 14 the ruler is introduced, but not as a substitute for the number line, which would help define fractions as numbers. Instead, the “whole” is 12 (inches) and  $\frac{1}{2}$  of this is 6 inches. Standard 3.3.A is not met in Grade 3 in the regular program, but in a very short Washington state supplement, A5, fractions are put on the number line and the ruler is also used as a substitute for the number line quite nicely.

Units 3 and 6 of Grade 4 work extensively with fractions, but still not as numbers except that in Session 18 of Unit 6 decimals are put on the number line.

In Grade 5, Unit 4, fractions come up again. In Session 11, fraction strips are used, but they are not used as part of the number line as fractions (fractional parts) of the strip. There are many different non-numerical representations of fractions and fraction addition is demonstrated in several of them. The closest things come to using the number line is the use of a ruler in Session 21, but this is for measurement, not for adding and subtracting.

Fractions continue in Unit 6. The ruler is used again in Session 3. This is a particularly nice model because the fraction strip and clock models all have a limit of 1, but the ruler has numbers on it that are greater than 1. The clock model of addition and subtraction is very nice as it can handle fractions whose denominators divide 60. Unfortunately, students are work with hours rather than numbers. In Session 6 there are mixed numbers and fractions greater than 1, but adding and subtracting is still done with models rather than numbers. In particular, the number line model for addition of fractions is not there. In Session 13 we see decimals and fractions put on a number line, including some decimals that are more than 1 put on a number line of length 2.

A proper numerical representation of adding and subtracting fractions seems to be missing. In addition, common denominators are never developed except for small “common” fractions. The general case is not dealt with.

For Grade 5 there are special Washington state supplements designed to cover the standards. In supplement A5 there are a number of problems adding and subtracting using the number line as a model (5.2.A). Unfortunately, the problem they use for demonstration purposes is done wrong. This is on page A5.5, “Fractions on the trail.” The example question is:

Marissa and her mom ran the first  $1\frac{1}{4}$  miles of the (2 mile\*) trail. They got tired, so they walked the rest of the way. What fraction of the trail did they walk?

\* shown in the sketch

There is a sketch, and explanation (in words) and an equation. The answer given is  $\frac{3}{4}$  mile. That is how many miles they walked, not the answer to the question that was asked: “What fraction of the trail did they walk?” The answer is  $\frac{3}{4}$  mile over 2 miles, or  $\frac{3}{8}$  (the miles cancel). Since this example problem amounts to the instruction, this is a serious flaw.

Standard 5.2.E about adding and subtracting fractions is the pinnacle of this thread and it is addressed in supplement A6. Students are not taught how to add and subtract fractions with the usual formula using a common denominator:

$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$ . Instead, rather than deal with big common denominators, they introduce the “least common denominator.” This is certainly mathematically correct, but the least common denominator is hard to find. Students are taught how to do this by writing down all of the multiples of each denominator until a common multiple is found. This is used to add and subtract fractions. They do examples of denominators 3 and 5, 6 and 8, and 4 and 10. This computational technique works well enough for the examples they work and for the problems they submit to the students. However, this standard is about adding and subtracting arbitrary fractions. The formula above would work very well and very quickly for the problem  $\frac{1}{17} - \frac{1}{19}$ , but *Bridges’* computational approach to finding the least common denominator would be a computational challenge that would take a great deal of time. Mathematically, it is insufficient.

The only word problems (5.2.H) involving the adding and subtracting of fractions come before adding and subtracting of fractions is mastered. The problems are as much about learning to add and subtract fractions as they are about solving word problems. That is not the intent of 5.2.H.

### Conclusions

The Grade 3 supplement gets fractions on the number line nicely (3.3.A). Representing adding and subtracting fractions as numbers, (5.2.A), is not done well. In the Grade 5 supplement where this is attempted the guiding example is

worked wrong, or, the question is asked wrong. Common denominators are not thoroughly developed, even in the supplementary material, so 5.2.E is not met. There are no word problems (5.2.H) following the completion of their work with adding and subtracting fractions. Students would not be prepared to go on.

### **Adding and subtracting fractions in Math Connects**

Fractions are introduced in Grade 2, Chapter 9, but not as numbers, as fractions “of” other things. In Grade 3, Chapter 13, fraction strips are introduced, but then they are put on the number line. This is very carefully done even in the student materials. At this stage students only look at fractions between 0 and 1 although the number line indicates that it goes off further in both directions.

In Grade 4, Chapter 13, fractions are well established as numbers on the number line, first as between 0 and 1 and then more arbitrarily, i.e. improper fractions and mixed numbers, which are also placed on the number line.

In Grade 5, Chapter 10, various representations for adding and subtracting fractions are given. When it comes to fractions with unlike denominators, the least common denominator is emphasized. This is taught through least common multiples in Chapter 9. This would qualify as a significant problem but Math Connects is saved by the note at the beginning of the study of the least common multiple on page 397:

Remember. You can always find a common multiple by finding the product of the numbers.

There are good word problems as well, for example, from page 454:

Warner lives  $9 \frac{2}{3}$  blocks away from school. Shelly lives  $12 \frac{7}{8}$  blocks away from school. How many more blocks does Shelly live away from school than Warner?

There are good multi-step problems as well (page 441):

A mosaic design is  $\frac{7}{15}$  red,  $\frac{1}{5}$  blue, and  $\frac{1}{3}$  yellow. What fraction more of the mosaic is blue and yellow than red?

### Conclusions

This thread is thoroughly covered. The only weakness is the emphasis on a least common denominator for adding and subtracting fractions and downplaying the easily accessible and obvious common denominator. Still, students should be prepared.

## **Summary Conclusions for Elementary School**

### **TERC Investigations**

The necessary components of whole number multiplication are there, but fluency with the standard algorithm is not developed.

In the area thread, the formulas are developed in a one-page supplement, but the height of a triangle is never explained.

Addition and subtraction of fractions using common denominators is not developed adequately.

### **Math Expressions**

The whole number multiplication thread is done extremely well.

The area thread is done extremely well.

The adding and subtracting fractions thread is also done extremely well.

It should be pointed out that almost all of the mathematics in these threads is in the student materials with the exception of the work on common denominators.

### **Bridges in Mathematics**

With the help of the Washington state supplement, this thread is well covered. It suffers somewhat from a lack of coherent goal to reach the standard algorithm. The thread is there, but it branches continuously.

The area thread suffers significantly. The regular program never defines area properly or arrives at formulas. The supplement includes the content, but there are problems. The derivation of the formula for the area of a parallelogram is mathematically incorrect and the derivation of the formula for the area of a triangle is inadequate.

There is a mathematical error in a major example used to illustrate a central point about the addition of fractions and common denominators are not developed. Given the scarcity of worked sample problems, every problem is critical. The thread is inadequately covered.

### **Math Connects**

The whole number multiplication thread is nicely done. The standard algorithm is more thoroughly dealt with than in the other programs even though the nice numerical model is missing.

The area thread is incomplete, lacking a formula for the area of a parallelogram and lacking any consideration of the area of a triangle.

The adding and subtracting fractions thread is nicely done except that the common denominator, although present, is downplayed.

All of the mathematics is very nicely presented and everything that is done is in the student materials.

## Final Conclusions

Judging by the three very important threads evaluated here, Math Expressions is the best program. Math Connects is very well done mathematically but is somewhat incomplete. These two programs are mathematically acceptable.

*Bridges in Mathematics* has mathematical errors and is incomplete. *TERC Investigations* has weaknesses in all three examined threads. Neither of these programs is mathematically acceptable.

## Middle School (6-8)

### Multiplication and division for fractions

Continuing the fraction thread from elementary school we will focus on:

6.1.D *Fluently and accurately multiply and divide non-negative fractions and explain the inverse relationship between multiplication and division with fractions.*

and the corresponding fraction part of the application standard

6.1.H *Solve single- and multi-step word problems involving operations with fractions and decimals and verify the solutions.*

Of course this all has to make sense so we need a good representation for fraction multiplication. It is difficult to represent fraction multiplication for anything but simple fractions so there has to be a logical connection and explanation that takes the student from the simple cases they can see with a representation to the general formula. There are multiple ways to do this, both geometric and algebraic (numerical), but it must be done.

Although certain cases of division with fractions can be represented, it is difficult to represent even simple situations such as  $\frac{1}{2} \div \frac{2}{3}$ . There are a number of ways to make sense of fraction division but bringing in the inverse relationship between multiplication and division can make it very easy. Any explanation must make sense not only of the simple example just given, but also of examples such as  $\frac{2}{9} \div \frac{3}{7}$ .

### Multiplication and division of fractions in Math Connects

In the student textbook, multiplying fractions starts on page 280 of Course 1. This moves on to dividing fractions and also covers mixed numbers, and ends on page 301.

Multiplication starts off multiplying  $\frac{1}{3}$  by  $\frac{1}{2}$  using a model of a square. First  $\frac{1}{2}$  of the square is shown and then  $\frac{1}{3}$  of that. It is easy to see  $\frac{1}{6}$  of the square in the end. Students work similar simple exercises and then  $\frac{3}{5}$  times  $\frac{2}{3}$  is

demonstrated using the same technique. More exercises are given. This model is very clear and, in these simple cases shows the product of two fractions very nicely.

Unfortunately, on the next page, page 282, students are told how to multiply two arbitrary fractions, “multiply the numerators and multiply the denominators”, with no further explanation except that it seems to work for fractions with very small denominators (2, 3, 4, 5, 6, and 8 are the only denominators used).

Next students learn how to multiply whole numbers by fractions. There is one nice model for  $\frac{3}{5}$  times 4, but the way it is explained, in general, is to treat a whole number  $n$  as a fraction,  $n/1$ .

There are a number of nice problems including some multi-step problems such as:

About  $\frac{1}{15}$  of a pint of blood is pumped through the human body with every heartbeat. If the average human heart beats 72 times per minute, how many quarts of blood are pumped through the human body each minute?

The multiplication of mixed numbers is done by converting them to improper fractions, using the formula and then converting the answer back to a mixed number.

Multiplication starts and ends nicely, but the transition from models to a general formula comes with no explanation.

Dividing fractions is introduced on page 291. The first example is a model to show how to divide 1 by  $\frac{1}{5}$ . Similar exercises are given. Next,  $\frac{3}{4}$  is divided by  $\frac{3}{8}$  using a nice model. Finally, a general argument is made that dividing a whole number by  $\frac{1}{2}$  is the same as multiplying by 2. From taking the reciprocal of  $\frac{1}{2}$  there is a leap, and the formula for dividing fractions is given: “To divide by a fraction, multiply by its reciprocal.” There is no further explanation. There are problems and mixed numbers are dealt with.

There is no attempt to explain the division formula other than the discussion of how it works for dividing whole numbers by  $\frac{1}{2}$ .

The above is from the student edition of the text. All we find in the Teacher Edition is a reiteration of the rules. There are no explanations, even for teachers.

### Conclusions

Multiplication of simple fractions is nicely modeled but no explanation for the general rule for multiplying fractions is given.

The only attempt to explain the division of fractions is for the cases where the answer is a whole number. No explanation is given for the rule for dividing fractions.

## Multiplication and division of fractions in Math Thematics

Fraction multiplication is done on pages 172-183 of Book 1. The area model (paper folding) is used to demonstrate  $1/3$  of  $2/3$ . Three exercises are worked by students and from these four examples students deduce the general rule for multiplying fractions (on page 175). The rule is not spelled out in the text at this point. Mixed numbers are then dealt with both by turning them into improper fractions and by using the distributive law.

On page 178 the rules for multiplying fractions and mixed numbers are stated clearly. No more explanation is given other than that it works for the simple examples that could be done by paper folding.

Nice problems are presented, including multistep problems such as:

One fifth of a farmer's corn crop is destroyed by hail. Later that summer,  $1/3$  of the remaining crop is eaten by insects. To file an insurance claim, the farmer needs to find what part of the total original crop was eaten by insects. The farmer says it is  $1/15$ . Is this correct? Explain.

Division of fractions is done on pages 345-356. Reciprocals are introduced and dividing by  $1/2$  is explained and shown to be the same as multiplying by 2. On a "Labsheet" (6A) students model some simple problems: 3 divided by  $1/4$ ,  $1\ 1/8$  divided by  $1/8$ , and  $2\ 1/4$  divided by  $3/4$ . The problems are worked after this using reciprocals. It is then stated (page 348) "you can use a reciprocal and multiplication to divide by a fraction." The examples to this point have all had whole number answers.

Next they move on to dividing 8 by  $3/4$  using the reciprocal rule but they ask the student to explain what the 10 means in the answer and what the  $2/3$  means. Other examples are given. General fractions and mixed numbers are then divided, culminating in a clear statement of the rules on page 352: division of fractions is done by multiplying by the reciprocal and division of mixed numbers is done by converting to improper fractions.

There are nice problems, a good multi-step problems is:

How many  $1/2$  inch by  $1/2$  inch square tiles are needed to make a boarder around the piece of paper? [The dimensions of the paper were given earlier.]

The above is from the student edition of the text. All we find in the Teacher's Edition is a reiteration of the rules. There are no explanations, even for teachers.

### Conclusions

Multiplication of simple fractions is nicely modeled but no explanation for the general rule for multiplying fractions is given.

The only attempt to explain the division of fractions is for the cases where the answer is a whole number. No explanation is given for the rule for dividing fractions.

### **Multiplication and division of fractions in Holt Mathematics**

Multiplication and division of fractions is done in pages 254-289 of Course 1. This is introduced first using whole numbers multiplied by fractions. They use the repeated addition model and also an area model to do this. The first example they use is 3 times  $\frac{1}{4}$ . As soon as they have modeled it they explain: "There is another way to multiply with fractions." The other way is to make 3 a fraction,  $\frac{3}{1}$ , and multiply the numerators and the denominators. There is no attempt at explanation. More examples using repeated addition are done and this is also done using the rule, but when one of the numbers is still a whole number turned into a fraction with 1 for the denominator.

On page 258 the program uses grids (areas) to model fraction multiplication for simple fractions such as  $\frac{1}{2}$  times  $\frac{1}{3}$  and  $\frac{2}{3}$  times  $\frac{1}{2}$ . Exercises are given. Mixed numbers are introduced for one factor.

On page 260 one last area model is given and then we are told "You can also multiply fractions without making a model." "Multiply numerators. Multiply denominators." The rule has been given. No explanation accompanies it. Mixed numbers are dealt with. Applications follow, such as:

The number of American bison has steadily declined throughout the years. Once, 20 million bison roamed the United States. Now, there are only  $\frac{1}{80}$  of that number of bison. Of those, only  $\frac{3}{125}$  roam in the wild. The number of American bison currently roaming in the wild is what fraction of 20 million? How many is that?

This is a nice multi-step problem but with various "facts" wrong. Bison population has varied tremendously, both up and down. It has not "steadily declined". After the introduction of human diseases in America, Indian populations declined and bison populations increased dramatically. In the 19<sup>th</sup> century they were killed off to near extinction but in the last 100 years have recovered to several hundred thousand.

On page 268 division by fractions is introduced. First, mixed numbers are divided by whole numbers. Then mixed numbers are divided by fractions when the result is a whole number. Area models are used for both. Reciprocals are introduced (270). At this stage it is pointed out that dividing a whole number by a whole number is the same as multiplying one number by the reciprocal of the other. This could have been the start of an explanation of fraction division, but it goes nowhere. The rule of dividing by multiplying by the reciprocal is established.



Pages 274 and 275 are devoted to solving simple equations such as  $\frac{5}{6}x = 4$ .

This could very easily be used to explain the rule for division of fractions but that is not done and in all of the examples one of the numbers is a whole number as well.

The above is from the student edition of the text. Almost all we find in the Teacher's Edition is a reiteration of the rules. There are no explanations, even for teachers. However, there is one little note on page 272: "A second method ... is to first rewrite the dividend and divisor using a common denominator." This could be turned into an explanation of division of fractions but it isn't.

Course 2 adds little. Even worse, on page 194, there is the blatantly false statement that "When you multiply a positive fraction by a positive fraction, the product is less than either factor." They are clearly assuming that all fractions are less than 1, but their own definition of fraction does not impose this limit.

### Conclusions

Multiplication of simple fractions is nicely modeled but no explanation for the general rule for multiplying fractions is given.

The only attempt to explain the division of fractions is for the cases where the answer is a whole number. No explanation is given for the rule for dividing fractions.

There is a section on solving simple equations that could have been turned into an explanation of division and in some ways comes close, but it is not taken advantage of in this way.

Both multiplication and division are broken down into small bits for easier digestion. For example, have one number be a whole number and only one be a fraction.

### **Multiplication and division of fractions in Prentice Hall Mathematics**

Multiplication and division of fractions is covered in pages 258-303 of Course 1. They begin by using the area model (folding paper) for fraction multiplication for simple cases such as  $\frac{1}{2}$  times  $\frac{3}{4}$ . After 4 very simple exercises students are asked to "Write a rule you can use to multiply two fractions without using a model." Another example is modeled ( $\frac{1}{2}$  times  $\frac{5}{6}$ ) and then the rule for multiplying fractions is stated, with no attempt to explain it. Fractions are multiplied by whole numbers by converting whole numbers to fractions. Good problems are worked:

At the movies, you eat all but  $\frac{1}{3}$  of a box of popcorn. Your friend eats  $\frac{2}{3}$  of what is left. Who eats more popcorn, you or your friend?

Although no computation is necessary to solve this problem, it is still a good problem because the student really has to pay attention to the meaning of what is said.

Mixed numbers are multiplied by changing them to improper fractions.

On page 271, fraction division is introduced using circular models for 3 divided by  $\frac{1}{8}$ . Dividing whole numbers by  $\frac{1}{2}$  is studied. The reciprocal is introduced. The rule is stated. No explanation is offered. Mixed numbers are divided.

On page 282 there begins a section on solving fraction equations. Equations such as  $\frac{5}{8}x = 6$  are solved. If 6 were a fraction, this almost gives an explanation of what it means to divide by  $\frac{5}{8}$  and how the reciprocal comes up. The connection is not made though and only whole numbers are used until some exercises at the very end on page 285 where fractions are used twice.

The above is from the student edition of the text. Almost all that is found in the Teacher's Edition is a reiteration of the rules. On page 258D there is the observation, not seen elsewhere yet, that "Multiplication and division are inverse operations that undo each other." This is in the teacher discussion about solving fraction equations but is not used anywhere. On page 261 there is a "Math Background" comment about the rule for multiplying two fractions: "This rule is, in fact, an algebraic theorem. That is, the rule can be justified by a logical argument that is supported by known algebraic properties." At least there is some acknowledgement that there is more to do, even if it is not seen as necessary to do for students (or teachers). No such generosity is exhibited towards division though. There are no explanations, even for teachers.

### Conclusions

Multiplication of simple fractions is nicely modeled but no explanation for the general rule for multiplying fractions is given.

The only attempt to explain the division of fractions is for the cases where the answer is a whole number. No explanation is given for the rule for dividing fractions.

There is a section on solving simple equations that could have been turned into an explanation of division and in some ways comes close, but it is not taken advantage of in this way.

The Teacher's Edition exhibits an awareness that these formulas could be explained even though no explanations are given. In addition, the Teacher's Edition does make the statement that multiplication and division are inverse operations, but makes no use of this fact.

### **Proportions**

After establishing control over the rational numbers, middle school is very much about rates, ratios, percents and proportions. Proportions will be reviewed with the focus on student's understanding well enough to work problems, so the following standard is examined:

*7.2.B Solve single- and multi-step problems involving proportional relationships and verify the solutions.*

Of course, to do this students need to be told what ratios, rates and proportions are and see a good development of representations for proportions. Cross products always show up here and this technique should be justified. We would like to see multi-step problems.

Proportions can be put at the center of a number of mathematical connections. To see how well this is done we will focus on the standard:

*7.2.F Determine the slope of a line corresponding to the graph of a proportional relationship and relate the slope to similar triangles.*

First, proportions need to be represented as linear graphs through the origin. To do this requires an understanding of slope. In order to show that slope is well defined, it is necessary to use the proportionality of similar triangles. This is required in order to show that a proportionality equation is a straight line. This standard connects up much of middle school mathematics.

### **Proportions in Math Connects**

Proportions are introduced in Course 1, Chapter 6, Lesson 3 (6-3), where clean definitions of proportional and proportion are given. Students are then taught how to tell if things are proportional or not. Good multi-step problems are given such as:

Rosalinda Saved \$35 in 5 weeks. Her sister saved \$56 in 56 days. Did each sister save proportionally the same amount of money? Explain.

Proportions are then set up in an equation with an unknown. Another example of a multi-step problem is:

Liliana takes 4 breaths per 10 seconds during yoga. At this rate, about how many breaths would Liliana take in 2 minutes of yoga?

Lesson 6-7 is about graphing a proportional relationship, but the way the lesson is set up with a graphing calculator does not require student understanding of the proportional relationship. One important way to look at proportions is as the graph of a line through the origin, which *Math Connects* does in lesson 3-7 of Course 2 where four points are plotted on the graph of  $y=2x+1$ . The program includes the following language:

Notice that all four points in the graph lie on the same straight line. Draw a line through the points to graph all solutions of the equation  $y=2x+1$ .

And:

An equation like  $y=2x+1$  is called a linear equation because its graph is a straight line.

Thus it is clear that a straight line gives all solutions and that all solutions give a straight line, but it is not clear why that it true. There are no explanations for either of these statements. For understanding, these statements both require explanations. Jumping ahead to Lesson 9-4 of Course 3 on slope, students are taught that they can compute the slope of a line using any two points on the line, but there is no attempt to explain why, i.e. why you always get the same slope no matter what two points you chose. The graph for proportions is done in Lesson 9-5, but under the name of direct variation rather than proportions. The connection between proportions and direct variation is made in a in a sidebar “study tip,” where it might easily be missed:

Notice that the graph of a direct variation, which is a proportional linear relationship, is a line that passes through the origin.

In Lesson 9-5 there is a non-rigorous attempt to show that the slope is the same no matter what two points you use. Unfortunately, it is not connected up with proportion problem solving even though proportions are used to do it.

Proportional and proportion are defined again in lesson 6-6 of Course 2 and in lesson 4-5 of Course 3. Cross products is introduced, explained and used both years. Simple proportional equations are set up and carefully solved. Many nice problems involving proportions are scattered throughout Courses 2 and 3 and these problems cover a variety of contexts. A typical example:

A 5-pound bag of grass seed covers 2,000 square feet. An opened bag has 3 pounds of seed remaining in it. Will this be enough to seed a 14-yard by 8-yard piece of land?

### Conclusions

Ratios, rates, proportional and proportion are properly defined (in all three years) and proportion problems are numerous, of varying types, and in all three years of the material. Cross products are explained properly. The connection to a proportionality constant and a graph is there but a little weak and definitely not emphasized, and the connection to slope and similar triangles is also there but even weaker.

### **Proportions in Math Thematics**

Math Thematics does not define ratio (Book 1, Module 6, Section 1 (6-1)). Instead, multiple **notations** for ratio are described, without defining the term: “The ratio of two numbers or measures can be written several ways: 4 to 6, 4:6,  $4/6$ .” This is how it can be written, but that is not a definition. This lack of clarity is compounded with the first example, which uses squares, typically used to

measure area, to measure the height of a small mug and a large mug. This seems to invite confusion between linear and square measurement. Ratios are formally defined in a subsequent lessons. (page 368)

A ratio is a special type of comparison of two numbers or measures. Ratios can be written in different ways. The order of the numbers in a ratio is important.

But this definition is not clear. “A special type of comparison” is not a definition and the glossary’s definition, “a type of comparison,” is also not a mathematical definition. Math Thematics builds on the undefined ratios to define rates in section 6-2.

By the time proportion is introduced in 6-4, the notation used for ratios is the same as that used for fractions, and with that unspoken assumption, proportion is defined properly. The first method used to determine if two ratios are in proportion is the cross product. There is a circle model of the first example  $16/10 = 24/15$  that shows 16 times 15 in one circle and 10 times 24 in the other circle. Although no explanation of what was done is given that seems easy enough to figure out. The real problem is that no reason is given as to why the equality should be preserved.

*Math Thematics* includes nice, but simple, problems, for example:

Suppose a school fundraiser makes a \$1.75 profit for every 6 rolls of wrapping paper sold. What is the profit on 256 rolls?

In Book 2, Section 2-2, a ratio is defined as “a comparison of two quantities by division.” Cross products, 5-3, are used but not explained.

Slope is introduced in 2-2 and students are asked (page 101) “would you get the same results if you used a different set of points to find the slope of the line for swimming?” The Teacher’s Edition explains that “The ratio of rise to run is the same anywhere along the line because the line’s steepness does not change.” Unfortunately this is circular reasoning since “steepness” means “slope.”

In Book 3, 3-3, students are asked when computing the slope “does it matter which points you choose?” The Teacher’s Edition explains “No; Sample response: We tried several different pairs of points and always got the same answer.” This is evidence, not proof. The Teacher’s Edition references Module 3, where there is a discussion of the slope-intercept form of the equation for a line, but it does not show that that the graph of such an equation really is a line.

In Book 3, 3-3, there are some graphs for rates, but the connection to proportions is not explicit.

### Conclusions

Book 1 sets up a confusing foundation for ratios, rates and proportions by not giving a definition of ratio. This is corrected in Book 2. Cross products are not explained.

There are proportion problems in limited number throughout the three years, although a lot of the problems are scattered throughout the book.

There is minimal connection of graphs to ratios and rates but nothing explicit about proportions, such as the constant of proportionality. Linear equations are not shown to produce graphs that are lines, and no reason is given for slopes of lines to be independent of points chosen to compute them. In all, the concept of proportions is underdeveloped.

### **Proportions in Holt Mathematics**

In Course 1, Chapter 7, there is a quickly corrected false start. In the chapter preview while talking about vocabulary, there is a statement, "Ratio can mean 'the relationship in quantity, amount, or size between two things.'" There is no indication of the nature of the relationship in this definition, but this is corrected two pages later with "A ratio is a comparison of two quantities using division." Notation is set up and equivalent ratios are studied. There is another example of mathematical sloppiness on the next page with, "A rate compares two quantities that have different units of measure." The proper definition can be found in the glossary, but not in the student or teacher materials.

Proportions are defined properly when they show up (on page 260 in an Exploration), but, unlike other words that are defined, it is not highlighted. Although students should be fluent with fractions at this point in time, all of the early exercises with proportions produce models.

Proportion is highlighted and defined in 7-3 of the regular text. Cross products are introduced, but not explained in the student text. In the Teacher's Edition there is an explanation, but not on the same page. There is good problem solving guidance, but the problems tend to be fairly simple. One of the better ones is:

Ursula is entering a bicycle race for charity. Her mother pledges \$0.75 for every 0.5 mile she bikes. If Ursula bikes 17.5 miles, how much will her mother donate?

Unfortunately, on the same page (364) there is also a badly phrased problem:

Given that the first term in a sequence is  $\frac{7}{2}$ , the second term is  $\frac{14}{4}$ , the fourth term is  $\frac{28}{8}$ , and the fifth term is  $\frac{35}{10}$ , find the value of the third term.

This is a poorly designed problem on several levels. First, there is insufficient information to determine what the third term should be as no rule for the sequence is given. Without a rule for the sequence, the third term could be anything. It is easy to surmise that the correct answer is that the third term is

21/6, but that doesn't make it a well-written problem. Furthermore, the question asks for the value of the third term and the desired answer is 21/6. The value of all of the terms is 3.5. The question should be written to force the desired answer. In the section called "multi-step" problems, the problem is:

You have \$100 in U.S. dollars. Determine how much money this is in euros, Canadian dollars, renminbi, shekels, and Mexican pesos.

This is actually a series of single step problems made easier because of the currency conversions graph associated with this problem

Course 1 is very limited with respect to proportions. The work with simple problem solving is nice and the work on indirect measuring and scale drawings is also nice.

In Course 2, Graphing linear functions (4-6) *Holt* states, "A linear equation is an equation whose graph is a line." The first example is  $y=2x+1$ , but it is never explained why its graph is a line. The clarification is that "Only two points are needed to draw the graph of a linear function." This is true, but it does not explain what a linear function is. The usual way would be to show that an equation such as  $y=2x+1$  has a graph that is a line. This requires some work with similar triangles, which is missing here. It is just assumed that certain types of equations have lines for graphs. Slope is introduced in 5-3 but there is no attempt to show how this connects to linear functions or lines. The slope is related to rates and we are told that "The graph of a constant rate of change is a line" but how and why this works is not explained.

In 5-1, ratios and rates are taken up again and proportions are done in a later section. In 5-5 we have a nice multi-step problem:

In June, a camp has 325 campers and 26 counselors. In July, 265 campers leave and 215 new campers arrive. How many counselors does the camp need in July to keep an equivalent ratio of campers to counselors?

Course 3 is thin on work with proportions, but what it does is introduce different types of problems that can be solved using proportions. Most problems are fairly direct, but there are a variety including the following:

Jacob is selling T-shirts at a music festival. Yesterday, he sold 51 shirts and earned \$191.25. How many shirts must Jacob sell today and tomorrow to earn a total of \$536.25 for all three days?

There is more on slope in Course 3, in 12-2, but again, it is missing clear explanations. For example:

Slope measures the rate of change in an algebraic relationship. Linear equations have constant rates of change. This means that the rate of change is always the same. This is shown in a graph by a straight line.

It has never been explained why the slope, or rate of change, is constant for a line. It has not been explained that this constant rate of change is shown by a graph being a straight line. It has not been explained that a constant rate of change gives rise to a straight line. Finally, in 12-5, there is a very short section on direct variation and the constant of proportionality.

### Conclusions

Rates are not properly defined when they are introduced. Proportion problems are thin and some are mathematically inappropriate, but there are also some good problems. Cross product is not explained in the student text. Linear equations, their graphs, and slopes are not done thoroughly. Proportions, slopes, and graphs are not linked up well through similar triangles and the proportionality constant, but direct variation does show up.

On the whole, mathematical concepts are introduced and used without adequate definitions or explanations.

### **Proportions in Prentice Hall Mathematics**

In Course 1, Chapter 7, ratios are defined, their notation is set up, and equivalent ratios are discussed. Along the way there is a most unusual problem for a math book:

Define *salad*. (Page 311)

Rates and proportions are defined. Cross products are introduced and used in Book 1, but not defined and explained until Book 2. Book 1, Teacher's Edition, does include an example that is explained in detail.

A straightforward setting up of a proportion is labeled a "challenge" problem:

You charge \$7 to baby-sit for 2 hours. Last night you earned \$17.50.  
How long did you baby-sit?

Most of the problems in Chapter 1 are similar to this.

In Course 2, Chapter 5, ratios, rates and proportions are taken up again. This is another "challenge" problem:

A bag contains colored marbles. The ratio of red marbles to blue marbles is 1:4. The ratio of blue marbles to yellow marbles is 2:5. What is the ratio of red marbles to yellow marbles?

Course 2 includes some work getting equations from proportions, and an example of a proportion from a graph. The connections are not made between these.



In 10-2, linear equations are defined to be those equations whose graphs are a line, but there is no attempt to show that the graph of equations such as their  $y=x+5$  actually give lines, they are just assumed to be linear equations.

Slope is introduced in 10-3. There is no attempt to show that slope is well defined, i.e. that no matter how you compute the slope of a line in the coordinate plane, you get the same answer. The Teacher's Edition has nothing in the way of explanation to contribute but it does at least state it: "The slope for any given line is constant."

In 10-3a, a Technology Activity Lab, includes, "can use a graphing calculator to explore the relationship between an equation and the slope of its graph." The equation used is  $y=2x+1$ , and this is the first equation used in the study of slope. Its slope is studied using a graphing calculator, although there is no conceivable benefit to doing this with a graphing calculator rather than by hand. It is a rather simple equation. Exercises consist of using the graphing calculator to compute more slopes of equations such as  $y=x+3$  and some exercises to find the slope without using the calculator.

In Course 3, in Activity Lab 4-3a there are two problems telling students "You can also use a graph to determine proportional and nonproportional relationships." This is not followed up anywhere though. There are more proportion problems in Course 3 but no clarification of slope and no constant of proportionality.

### Conclusions

Definitions are correct. Cross products are not explained in the first year but are in later years. There are a variety of problems but none are particularly difficult. There is essentially no connection between proportions and graphs, and slope is not shown to be well defined and linear equations are not shown to give lines as graphs.

### **Conclusions for multiplication and division of fractions**

In the world of mathematics education where such a premium is placed on conceptual understanding, it is disappointing to see the total lack of explanation of the rules for the multiplication and division of fractions. Formulas are given and taught by rote when they could easily be given clear, short, explanations. For example, if we know how to multiply fractions and we want to divide, we just use the inverse nature of multiplication and division. If  $A$  and  $B$  are fractions and we want to compute  $A/B = C$ , the inverse nature of division and multiplication says this is equivalent to  $A = BC$ . It is now easy to see that multiplying  $B$  by its reciprocal will give 1 so if we multiply the equation by the reciprocal of the fraction  $B$ , the right side is the quotient that we are after,  $C$ , and the left side is just  $A$  times the reciprocal of  $B$ . Two of the programs, Holt Mathematics and Prentice Hall Mathematics, come close to this, after division has been done by rote, when they solve equations of the sort  $A = BX$  by multiplying both sides by the reciprocal of  $B$ . Usually though, their  $A$  is a whole number. All four programs

fail to elucidate the multiplication and division of fractions but we would rate these two slightly above the other two, Math Connects and Math Thematics, in this regard.

None of these programs prepares students to go on in mathematics with any understanding of the operations. .

### **Conclusions for proportions**

Proportions bring up several issues. First, are the basic definitions of ratio, rate, and proportion even there? Math Connects and Prentice Hall both do well. Holt fails to define rates the first year and, most problematic, Math Thematics fails to define ratio in Grade 6. Second, cross products are used by all texts and are only justified the first time they are taught in Math Connects . Prentice Hall does so a year after they are introduced and Math Thematics and Holt do not explain them in the student texts.

Third, the quantity, variety, and quality of problems varies from program to program, but Math Connects is clearly the leader. Fourth, connecting proportion to the constant of proportionality and graphs is fairly weak even in Math Connects, where it is done best. Direct variation does show up in Holt. Math Thematics demonstrates a vague awareness of an issue here and Prentice Hall is at the bottom.

Using these four criteria - definitions, cross products, problems, and graphing - we find that Math Connects is head and shoulders above the others. Prentice Hall and Holt are tied and Math Thematics is clearly at the very bottom.

### **Final Conclusions for Middle School**

Overall, Math Connects ranks first, Prentice Hall and Holt tie for the middle place, and Math Thematics is clearly at the bottom.

Fractions are very important. None of these programs does fractions well enough to justify their use in a classroom where students are expected to understand what they are doing instead of just learn by rote. Grade 5 Math Expressions does a better job at multiplication and division of fractions than any of these programs does. It is not that difficult to explain why multiplication and division of fractions is done with the formulas presented in all of these programs, but none of them bothers. Ratios, rates and proportions are completely dependent on a student's understanding of fractions, so, even if they are done well, if the foundation of fractions isn't there, it isn't likely to matter.